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Reciprocal Symmetry and Equivalence Between Relativistic and Quantum Mechanical Concepts

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We have defined slowness (or reciprocal velocity) corresponding to velocity (v) as cc/v, where c is the speed of light and v is the corresponding velocity. Velocity and slowness are images of each other. Reciprocal symmetric law of addition of velocities fulfils the requirement that the sum (or the difference) of velocities remains unchanged if velocities are replaced by the corresponding slownesses. We have postulated reciprocal symmetry, which states that every valid statement has an equally valid (reciprocal) image statement. The postulate has allowed us to derive.

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I. INTRODUCTION

We have already observed a reciprocal symmetry and correspondence between quantum mechanics and special relativity[1]. In this paper we intend to study in detail the relation between the Principle of Superposition of Einsteins law of addition of velocities. This we shall study in section II. In section III we shall study the how Reciprocal Symmetry relates superposition of Eigenfunctions to relativistic addition of velocities. We shall develop reciprocal symmetric number system, and study how it is related to the results of the preceding parts in the final section IV.

II. VELOCITY EIGENFUNCTION AND RELATIVISTIC ADDITION OF VELOCITIES

We define a slowness, s, as the reciprocal conjugate of velocity, v, by the relation $sv = -i\hbar$. Like a momentum operator[2] we then define a velocity operator and find the corresponding Eigenfunction. Superposition of two Eigenfunctions corresponding to velocities u and v gives the Eigenfunction corresponding to $u+v$. This is Galilean addition of velocities.
Einstein's postulate sets an upper limit, c, to v. This implies a lower limit to s. We call it $q = \frac{-i\hbar}{c}$. We replace the differential velocity operator by the corresponding to finite difference operator and we set q as the lower limit of s. Superposition of velocities Eigenfunctions should give Einstein's law of addition of velocities. We shall study this in section 2.

A. Reciprocal Velocity and Velocity Operator

We define slowness, s, as the reciprocal conjugate of v

$$s = \frac{-i\hbar}{v} \tag{1}$$

And q is the reciprocal conjugate of, c, the speed of light

$$q = \frac{-i\hbar}{c} \tag{2}$$

velocity operator is

$$-i\hbar\frac{\delta}{\delta s} \tag{3}$$

The Eigenvalue equation is

$$-i\hbar\frac{\delta}{\delta s}\psi = v\psi \tag{4}$$

B. Superposition of Continuous Eigenfunctions and Galilean Addition of Velocities

Eigenfunction $\psi(v) = \exp(i\frac{s\cdot v}{\hbar})$ gives the Eigenvalue v.
Superposition of Eigenfunctions gives the resultant Eigenfunction

$$\psi(u)\psi(v) = \psi(u+v) \tag{5}$$

$(u+v)$ is the Galilean sum of velocities u and v.

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C. Discrete Eigenfunctions and Lorentz-Einstein Addition of Velocities

We now replace the continuous Eigenvalue equation by the corresponding to discrete equation

$$-i\hbar \frac{D}{\partial D(s,q)}\phi = v\phi \tag{6}$$

Where

$$\frac{D}{\partial D(s,q)}\phi = V(v)\frac{\phi(s+q)-\phi(s-q)}{2q} \tag{7}$$

We require

$$\left.\begin{array}{l} \frac{D}{D(s,q)} \to \frac{\partial}{\partial s}, \\ V(v) \to v, \\ \phi \to \psi, \end{array}\right\} \tag{8}$$

as $q \to 0$

We also require that (7) remains invariant under the change $q \to -q$. The relation between negation and reciprocation is studied in Appendix.

This requirement implies the requirement

$$\left.\begin{array}{l} V(v) \to V(v), \\ \phi \to \phi, \end{array}\right\} \tag{9}$$

as $q \to -q$

Our requirements are fulfilled if

$$V(v) = \frac{v}{\sqrt{1+(\frac{qv}{\hbar})^2}} = \frac{v}{\sqrt{1-(\frac{v}{c})^2}} \tag{10}$$

and

$$\phi(v) = \left(\frac{1+i\frac{qv}{\hbar}}{\sqrt{1+(\frac{qv}{\hbar})^2}}\right)^{\frac{s}{q}} = \left(\frac{1+\frac{v}{c}}{\sqrt{1-(\frac{v}{c})^2}}\right)^{\frac{s}{q}} \tag{11}$$

Relation (7) has been used to get the r.h.s of the above relations.

Superposition of 2 Eigenfunctions gives

$$\phi(u)\cdot\phi(v) = \phi(w) = \left(\frac{1+i\frac{qw}{\hbar}}{\sqrt{1+(\frac{qw}{\hbar})^2}}\right)^{\frac{s}{q}} \tag{12}$$

Where

$$w = \frac{u+v}{1-u\cdot v(\frac{q}{\hbar})^2} = \frac{u+v}{1+\frac{uv}{c^2}} \tag{13}$$

This is Lorentz-Einstein sum of velocities[3]

III. RECIPROCAL SYMMETRY IN QUANTUM MECHANICS AND SPECIAL RELATIVITY

A. Generalized Reciprocal and Negative

Reciprocal

$$a \oplus_\circ b = \frac{a+b}{1+ab} \tag{14}$$

[Compare (14) to (31) when $\phi = 0$]. We observe that the sum (14) is reciprocal symmetric i.e.

$$\frac{1}{a} \oplus_\circ \frac{1}{b} = a \oplus_\circ b \tag{15}$$

We also observe that the product

$$A \oplus_1 B = A \cdot B \tag{16}$$

is symmetry under negation i.e.

$$(-A) \oplus_1 (-b) = A \oplus_1 B \tag{17}$$

Introducing the reciprocal operator R, which takes a to its reciprocal $R(\phi,a)$ we write (for $c = 1$, see (30))

$$R(\phi=0,a) = \frac{1}{a} and R(\phi=1,a) = -a \tag{18}$$

We can write (15) and (17) under the same general relation

$$R(\phi,a) \oplus_\phi R(\phi,b) = a \oplus_\phi b \tag{19}$$

$\phi = 0$ in (19) will give (15), while $\phi = 1$ in (19) will give (17).

[For mathematical details see B11 of IV F]

B. Reciprocal Symmetry and Shift of Origin

We observe that (12) and (13) remain invariant under general reciprocal transformations

$$-\phi(u)\cdot-\phi(v) \tag{20}$$

and

$$w = \frac{u+v}{1+\frac{uv}{c^2}} = \frac{\frac{c^2}{u}+\frac{c^2}{v}}{1+\frac{(\frac{c^2}{u}\cdot\frac{c^2}{v})}{c^2}} \tag{21}$$

Choosing $\frac{s}{q} = 2$ in (11) we get

$$\phi(v) = \left(\frac{1+\frac{v}{c}}{\sqrt{1-(\frac{v}{c})^2}}\right)^2 = \frac{1+\frac{v}{c}}{1-\frac{v}{c}} \tag{22}$$

Therefore

$$\phi(\frac{c^2}{v}) = -\phi(v) \tag{23}$$

Using definitions (12), (19) and also introducing the notation $\phi(v) = \phi'(\frac{v}{c})$

$$\phi'(R(\phi = 0, \frac{v}{c})) = R(\phi = 1, \phi') \tag{24}$$

Therefore Einstein's law of addition of velocities and velocities Eigenfunctions demonstrate the same reciprocal symmetry.

Shift of Origin

Transformation of number a_0, measured in the system in which the neutral number is 0, to a_0, in the system in which the neutral number is 1, is given by transformation relation (28).

$$a_\phi = \frac{a_0 + \phi}{1 - a_0 \cdot \phi} \tag{25}$$

for $\frac{s}{q} = 2$ and $a_0 = \frac{v}{c}$ we have

$$\phi = a_1 = \frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \tag{26}$$

IV. RECIPROCAL SYMMETRIC REAL NUMBER SYSTEM

A. Axioms of Real Number System

Consider the Field Axioms[4] satisfied by the members of the set, R, of real numbers,

A1. $a + b = b + a$

A2. $(a + b) + d = a + (b + d)$

A3. There exists a 0 in R such that $a + 0 = a$ for all a in R

A4. For every a there is a N(0,a) in R such that $a + N(0,a) = 0$

and

A'1. $a.b = b.a$

A'2. $(a.b).d = a.(b.d)$

A'3. There exists a 1 in R such that $a.1 = a$ for all a in R

A'4. For every a there is a N(1,a) in R such that $a + N(1,a) = 0$

We observe that the two sets of axioms, shown above, are essentially the same except the difference in 0 and 1 in A3 and A'3.

B. Axioms of Extended Real Numbers

We extend the real number system by including 2 elements $+\infty$ and $-\infty$ to the set of real numbers. This set, E, will be called the set of extended real numbers[4]. The following set of axioms is included.

E1. $a + \infty = \infty$

E2. $a - \infty = -\infty$

E3. $a.\infty = \infty, if a > 0$

E4. $a. - \infty = -\infty, if a \neq \infty$

For all real number a

E5. $\infty - \infty$ is not defined

E6. $0.\infty$ is not defined

C. Generalized Notation

Introducing the notation $+ = \oplus_0$ and $\cdot = \oplus_1$ the above sets of axioms may be rewritten as

A1. $a \oplus_0 b = b \oplus_0 a$

etc. and

A'1. $a \oplus_1 b = b \oplus_1 a$

Introducing the unspecified number ϕ which, may be allowed to take values $\phi = 0$ and $\phi = 1$, the above sets of axioms may be written as

A"1. $a \oplus_\phi b = b \oplus_\phi a$

A"2. $(a \oplus_\phi b) \oplus_\phi d = a \oplus_\phi (b \oplus_\phi d)$

A"3. There exists a $a \oplus_\phi \phi = a$ in R such that $a \oplus_\phi \phi = a$

A"4. There exists a $N(\phi, a)$ for every a in R such that $a \oplus_\phi N(\phi, a) = \phi$

The set A" of axioms is just the sets A and A' rewritten. To achieve a real generalization, we re-enunciate the axioms as below.

D. Generalization of Addition to any Neutral Number ϕ

B1. It is possible to define $a \oplus_\phi b$ (depending on an arbitrarily chosen ϕ in R) for any a and b in R such that $a \oplus_\phi b$ belongs to R and that $a \oplus_\phi b = b \oplus_\phi a$

B2. $(a \oplus_\phi b) \oplus_\phi d = a \oplus_\phi (b \oplus_\phi d)$

B3. There exists a $a \oplus_\phi \phi = a$ in R such that $a \oplus_\phi \phi = a$

B4. There exists a $N(\phi, a)$ for every a in R such that $a \oplus_\phi N(\phi, a) = \phi$

E. Reflection Axioms

B5. Corresponding to every a in R there exists, depending upon an arbitrarily chosen $c \neq \phi$, a $R(\phi, a)$ in R

B6. $N(R(\phi, a)) = R(N(\phi, a))$

B7. $R^2(\phi, a) = a$ and $N^2(\phi, a) = a$

B8. $N(\phi, \phi) = \phi$

B9. $R(\phi, c) = c$

B10. $N(\phi, a) \oplus_\phi N(\phi, b) = N(\phi, a \oplus_\phi b)$

Theorem 1: $a \oplus_\phi N(\phi, a) = \phi$ Theorem 1 (Proof is given in IV J 1) above is Axiom B4. Therefore, it is not necessary to put B4 as a separate axiom. We keep it, nevertheless, for comparison with axioms A and A'.

F. Symmetry Axiom

B11. $R(\phi, a) \oplus_\phi R(\phi, b) = a \oplus_\phi b$

Note: B11 contains the equivalent of E1 (see Theorem 3).
Theorem 2: $a \oplus_\phi R(\phi, b) = R(\phi, a \oplus_\phi b)$
Proof is given in IV J 2
Theorem 3: $a \oplus_\phi c = c$
Proof is given in IV J 3

G. Multiplication Axioms

Greek letters α, β, γ etc. stand for elements of the familiar set, F, of real numbers extending from $-\infty$ to $+\infty$. Since there is no risk of confusion, we shall write $\otimes$ for $\otimes_\phi$ and $\oplus$ for $\oplus_\phi$, omitting the subscript ϕ.

B12. $(\alpha \otimes a)$ belongs to R for every α in F and a in R

B13. $\alpha \otimes (a \oplus b) = (\alpha \otimes a) \oplus (\alpha \otimes b)$

B14. $(\alpha \otimes a) \oplus (\beta \otimes a) = (\alpha + \beta) \otimes a$ where + is the familiar addition

B15. $\alpha \otimes (\beta \otimes a) = (\alpha \cdot \beta) \otimes a$ where '.' is the familiar multiplication

B16. $(-\alpha) \otimes a = \alpha \otimes N(a) = N(\alpha \otimes a)$ where $-\alpha$ is the familiar negative of α

B17. $0 \otimes a = \alpha \otimes \phi = \phi$.

Axioms B12-B17 also ensure consistency between Homogenous Number System and the familiar number system.

Undefined Quantities

U1. $c \oplus N(c)$ is not defined

U2. $0 \otimes c$ is not defined

U3. $\infty \otimes \phi$ is not defined

In the above list, U1 corresponds to E4. U2 and U3 correspond to E5.

H. Measures and Counts

The numbers we are talking about in earlier subsections till IV F, we shall call measures. On the other hand, the numbers for which we are employing Greek alphabets in section IV G, we shall call counts.

I. Isomorphic Transformations

Let α_ϕ and α_θ be the numbers which represent the same physical quantity; the first in the system in which ϕ is the neutral number, the second in the system in which θ is the neutral number (the origin). a_0 will, then, be the number in the familiar number system measured from 0. The transformation relation relating two numbers is

$$a_\phi = \frac{(1 + \theta \cdot \phi) \cdot a_0 - (\theta - \phi)}{(1 + \theta \cdot \phi) + a_\theta \cdot (\theta - \phi)} \tag{27}$$

Transformation to a_ϕ from the familiar number a_0 will be

$$a_\phi = \frac{a_0 + \phi}{1 - a_0 \cdot \phi} \tag{28}$$

J. Representations

$$N(\phi, a) = \frac{(\phi - a) + (1 + a \cdot \phi) \cdot \phi}{(1 + a \cdot \phi) - (\phi - a)\phi} \tag{29}$$

$$R(\phi, a) = \frac{(c - \phi)^2 (1 + a \cdot \phi) + \phi (1 + c \cdot \phi)^2 (a - \phi)}{(1 + c\phi)^2 (a - \phi) - (c - \phi)^2 (1 + a \cdot \phi)\phi} \tag{30}$$

$$a \oplus b = \frac{y + \phi}{1 - y\phi}, \tag{31}$$

where

$$y = \frac{\frac{a-\phi}{1+a\phi} + \frac{b-\phi}{1+b\phi}}{1 + \left(\frac{1+c\phi}{c-\phi}\right)^2 \frac{a-\phi}{1+a\phi} \frac{b-\phi}{1+b\phi}} \tag{32}$$

$$\alpha \otimes a = \frac{\left(\frac{c-\phi}{1+c\phi}\right) \frac{w^\alpha - 1}{w^\alpha + 1} + \phi}{1 - \left(\frac{c-\phi}{1+c\phi}\right) \frac{w^\alpha - 1}{w^\alpha + 1} \phi}, \tag{33}$$

where

$$w = \frac{\frac{c-\phi}{1+c\phi} + \frac{a-\phi}{1+a\phi}}{\frac{c-\phi}{1+c\phi} - \frac{a-\phi}{1+a\phi}} \tag{34}$$

Special cases
When $\phi = 0$ and $\phi = 1$ respectively, the general negatives are

$$\left.\begin{array}{l} N(\phi = 0, a) = -a, \\ N(\phi = 1, a) = \frac{1}{a} \end{array}\right\} \tag{35}$$

The general reciprocals are: when $c = 1$ it becomes

$$\left.\begin{array}{l} R(\phi = 0, a) = \frac{1}{a}, \\ R(\phi = 1, a) = -a \end{array}\right\} \tag{36}$$

$$R(0, a) = \frac{1}{a} \tag{37}$$

In this case the sum $\oplus_\phi$ becomes $\oplus_0$ and

$$a \oplus_0 b = \frac{a + b}{1 + \frac{a.b}{c^2}} \tag{38}$$

(38) has the form of Lorentz-Einstein law of addition of velocities. With $\phi = 0$ and $c \longrightarrow \infty$ the multiplication becomes

$$\alpha \otimes_0 a \longrightarrow \alpha \cdot a \tag{39}$$

With $\phi = 1$ and $c = 0$, in this case negation becomes

$$N(\phi, a) = N(1, a) = \frac{1}{a} \tag{40}$$

The reciprocal is

$$R(\phi, a) = R(1, a) = -a \tag{41}$$

The sum becomes

$$a \oplus_1 b = a \cdot b \tag{42}$$

1. Addition with General Negative

Theorem 1:

$$a \oplus_\phi N(\phi, a) = \phi \tag{43}$$

When there is no risk of ambiguity, we shall put $N(\phi, a) = N(a)$ and $R(\phi, a) = R(a)$. With this notation we rewrite the theorem
Theorem:

$$a \oplus N(a) = \phi \tag{44}$$

Proof: Let

$$a \oplus N(a) = y \tag{45}$$

Using B10

$$N(a) \oplus N^2(a) = N(y) \tag{46}$$

Using B7 and B1

$$a \oplus N(a) = N(y) \tag{47}$$

Comparing (45) and A(1,3)

$$y = N(y) \tag{48}$$

Therefore

$$y = \phi, or, R(\phi) \tag{49}$$

Two definitions of $\oplus$ are possible. One will give $a \oplus N(a) = \phi$. The other one will give $a \oplus N(a) = R(\phi)$. We shall use the notation $\oplus_\phi$ when it gives ϕ. We shall use the notation $\oplus_{R(\phi)}$ when the neutral number is $R(\phi)$.

2. Addition with General Reciprocal

Theorem:

$$a \oplus R(b) = R(a \oplus b) \tag{50}$$

Proof: Consider the sum

$$N(b) \oplus d = a \tag{51}$$

By B11 and B7

$$N(b) \oplus d = a = R(N(b)) \oplus R(d) = a \tag{52}$$

Therefore, using B4 and $N(R(N(b))) = R(b)$ right hand part of (52) gives

$$a \oplus R(b) = R(d) \tag{53}$$

Again using left hand part of (52) and $N^2(b) = b$, we have

$$a \oplus b = d \tag{54}$$

Comparing (53) and (54)

$$a \oplus R(b) = R(a \oplus b) \tag{55}$$

3. *Reciprocal Symmetric Analogue of Einstein's Postulate*

Theorem:

$$a \oplus c = c \quad (56)$$

Proof: Let

$$a \oplus c = d \quad (57)$$

Using B11 and B9

$$R(a) \oplus c = d \quad (58)$$

Also using (50) and $R(c) = c$

$$R(a) \oplus c = R(d) \quad (59)$$

Therefore,

$$R(d) = d \quad (60)$$

Therefore,

$$d = c \quad (61)$$

Therefore,

$$a \oplus c = c \quad (62)$$

V. RESULTS AND DISCUSSION

We have defined, section II, reciprocal conjugate variables, and have been able to derive relativistic law of addition of velocities from superposition of discrete wave functions. Then, in section III, we have shown that both quantum mechanics, principle of superposition and relativistic law of addition of velocities satisfy the same generalized reciprocal symmetry. Finally, in section IV, we have given algebraic details of reciprocal symmetric real number system.

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